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Dimensionality analysis of multiparticle production at high energies

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An algorithm for the analysis of multiparticle final states is offered. By the Renyi dimensionalities, which were calculated according to experimental data, either the hadron distribution over rapidity intervals or the particle distribution in an N-dimensional momentum space, we can judge the degree of correlation of particles, separate the momentum-space projections and areas where probability measure singularities are observed. The method is tested in a series of calculations with samples of fractal object points and with samples obtained by means of different generators of pseudo- and quasi-random numbers.

1. Phenomenological description of multiparticle production

The significant increase of information about multiparticle final states produced in particle collisions with higher and higher energies makes it urgent to develop non-traditional methods of analysis of experimental data. From the parameters of the detected particles one can construct many joint and conditional probability distributions which are much more informative than the averaged characteristics [1].

Since the general theory of strong interactions is not yet complete, a phenomenological approach to ultra-high-energy collisions is widely used. One of the first theoretical generalizations of multiparticle production processes is KNO scaling, which predicts that at sufficiently high energies the distribution of hadron multiplicity P_n obeys the scaling

$$P_n\langle n\rangle = \Psi(z), \quad z = n/\langle n\rangle, \tag{1}$$

where P_n is the probability to observe n hadrons in the final state, and $\langle n\rangle$ is the mean multiplicity at a given energy.

Since the Poisson distribution

$$P_n = \langle n\rangle^n \,\mathrm{e}^{-\langle n\rangle}/n! \tag{2}$$

describes the hadron multiplicity badly, it was proposed to use the negative binomial distribution and the Bose–Einstein distribution, which supposes the presence of k independent random sources with the same intensity:

$$P_n^{\langle k\rangle}\langle n\rangle = \Psi_k(z) = k^k z^{k-1}\,\mathrm{e}^{-kz}/(k-1)!. \tag{3}$$

Carruthers has shown [2] that $\Psi_2(z)$ describes the ISR and SPS data well.

Though the description of the nature of random sources meets difficulties, it has recently been possible, using the Bose–Einstein correlations, to estimate the size of hadron sources [3]. The source size in p–$\bar{\mathrm{p}}$ collisions did not change when the energy changed from 0.9 to 2.2 TeV in the c.m.s. (as was to be expected, if the KNO scaling was satisfied) and was in a linear dependence with the charge density in the pseudorapidity bin $(\Delta n/\Delta\eta)$:

$$R_{\mathrm{Fermi}} = 0.59 \pm 0.05(\Delta n/\Delta\eta). \tag{4}$$

Recently the particle distribution in rapidity windows has become the object of great attention. Large fluctuations in some rapidity bins, which were found in experiments at colliders and in cosmic-ray experiments [4], could not find any description in the frame of earlier suggested phenomenological mechanisms. The conclusion was drawn that the large fluctuations in the rapidity distributions reflect non-trivial fluctuations of hadronic matter during collisions.

Until how the instrument of investigation of non-trivial rapidity correlations has been the study of the dependence of normalized moments of the rapidity distributions on the size of the rapidity bin [5]. Several modifications of the moments method are suggested:

$$C_q = \langle n^q \rangle / \langle n \rangle^q, \quad q = 1, 2, \ldots, \tag{5}$$

$$C'_q = \langle (n - \langle n \rangle)^q \rangle / \langle n \rangle^q, \tag{6}$$

$$C''_q = \langle n(n-1) \cdots (n-q+1) \rangle / \langle n \rangle^q, \tag{7}$$

where q is the order of the normalized momenta and $\langle\ \rangle$ means averaging over the rapidity bins.

Let us write down a more detailed expression of a normalized moment,

$$C_q(M) = \frac{1}{M} \sum_{m=1}^{M} n_m^q / \langle n_m \rangle^q, \tag{8}$$

where M is the number of equal rapidity bins $\delta_y = \Delta/M$, Δ usually is the interval $(-2, 2)$, i.e. $\delta_y = 4/M$, n_m is the number of hadrons falling into the mth bin, and $\langle n_m \rangle$ is the average bin population of events with multiplicity n.

Let us consider, following ref. [6], how the normalized moments behave assuming first absence of correlation and then very strong correlation. Consider the uniform bins distribution: $n_m = N/M$, $m = 1, \ldots M$. It is easily seen that for all q, $F_q(M) = 1$. And if all the hadrons have fallen into the same bin, $n_m = N$ for some $m = r$, and $n_m = 0$ for the rest of m, then

$$C_q(M) = M^{q-1}, \tag{9}$$

i.e. at an extremal fluctuation the moments significantly increase with the number of bins. That is why the moments method sometimes is called a magnifier for exposure of non-uniformities. Rewriting eq. (9) in a somewhat different form and taking its logarithm gives

$$C_q(M) = (\Delta/\delta_M)^{q-1}, \tag{10}$$

$$\ln C_q(M) = -(q-1) \ln \delta_M + (q-1) \ln \Delta. \tag{11}$$

The moments logarithm depends linearly on the logarithm of the bin size. A random quantity with such a behaviour is called *intermittent* and the factor multiplying the logarithm of the bin size is called the *index of intermittency*. An intermittent random quantity in a sense is the opposite of a Gaussian one, for which a considerable deviation from the average value is very improbable.

If even after averaging over all the events (events with both the same and different multiplicity can be averaged), the scaling relation

$$\ln\langle C_q(M) \rangle = -\lambda_q \ln \delta_M + g_q \ln \Delta \tag{12}$$

is satisfied, then the physical process investigated is characterized by *intermittency*.

It is obvious that the experimental growth of normalized moments, revealed in a wide energy range of hadronic and leptonic collisions, is a new main characteristic of multiparticle production, which emphasizes the role of very short-range correlations compared with the usual short-range ones responsible for resonance production.

The first phenomenological mechanism describing the behaviour of factorial moments was the hypothesis of the existence of two types of sources: *luminary*, with a regular signal distribution, and *turbulent*, which is characterized by chaotic bursts [7]. When colliding, the parton, passing through and interacting in hadronic matter, enters high-density regions (narrow channels), emits many particles, also passes through low-density regions (wide channels) and uniformly emits few particles. In such an interpretation, the main attention is drawn to the very complicated trajectory of the partons wandering in the hadronic matter [8]. But we believe a much more natural way to interpret the anomalous behaviour of normalized moments is based on the hierarchy (self-similarity) of the processes of multiparticle

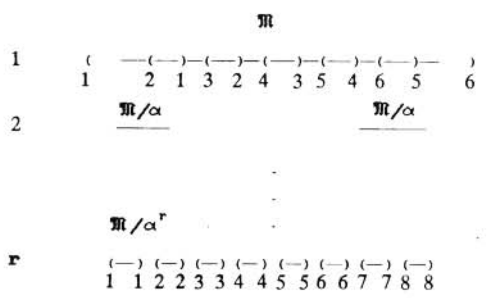

Fig. 1. Self-similar cascade decay of a particle with mass m. On the rth step of the cascade development there are $2r$ particles with mass m/α^r.

production and on the notion of fractal (multifractal) dimensionality, closely connected with self-similarity.

Relations like (12) are a consequence of self-similarity in the structure studied and give ground to carry out a dimensionality analysis. A dimensionality analysis means revealing in a $3N$-dimensional momentum space (or in a one-dimensional rapidity space) lower-dimensional regions where the events are grouped.

At present a number of simulations of quark–gluon cascade development in hadronic matter [9,10] are available. The updating of the LUND program based on the idea of parton–hadron duality [11,12] led to the realization that the unusual behaviour of normalized moments is due to the QCD cascade [12,13].

Before going on to the fractal analysis formalism, we shall show how a non-integer fractal dimensionality can arise in the simplest cascade process of the decay of a particle of mass m [14] (see fig. 1).

On each self-similarity step of the cascade the mass decreases by a factor of $1/\alpha$, with $\alpha \geq 2$. ($\alpha = 2$ if final-state particles are produced with zero kinetic energy). On the rth step of the cascade we have $2r$ particles with mass $(1/\alpha)^r m$. The masses of the particles obtained as a result of the cascade constitute the metric set $\mathscr{H}$.

Let us show that at the beginning of the cascade process the topological dimension $d_T \mathscr{H} = 1$, and then later $d_T \mathscr{H} < 1$.

The topological dimension is equal to $\mathscr{T}$, if it is possible to enter the finite open coverage of the multiplicity $\leq \mathscr{T} + 1$ into any finite open coverage of the set $\mathscr{H}$, and if there exist such finite open coverages of $\mathscr{H}$ into which it is impossible to enter finite open coverages of multiplicity $\mathscr{T} + 2$. The coverage multiplicity is the maximum number of coverage elements containing common points of the set $\mathscr{H}$ [15]. For our example, the possibility of entering coverages of multiplicity 2 into any open coverage of $\mathscr{H}$ is a necessary condition for the dimension to be equal to unity.

The open intervals 1–1, 2–2, ..., in fig. 1 form the finite open coverage of the set $\mathscr{H}$. As can be seen from this figure, it is possible to enter the coverage of multiplicity 2 for the incident particle – it is enough to take somewhat shorter intervals of coverage and they will also intersect, i.e. the multiplicity is 2; and for the rth step of the cascade it is impossible, since the intersecting intervals cannot be embedded in the non-intersecting ones.

2. The technique of dimensionality analysis

Cascade processes, which are frequent in high-energy physics, are due to some characteristic dimensionality. But, in contrast to the ideal self-similar cascades of geometric figures (e.g. Serpinski's carpet), in real physical systems there are possible deviations from self-similarity and, first of all, they contain not a single, but several characteristic scales connected with some dimensionality. The main goal of the dimensionality analysis is to reveal these dimensionalities and to relate them to the dynamic mechanisms responsible for their production.

There exist many different definitions of dimensionality. The following definition can easily be generalized to a non-integer case,

$$d_F = -\lim_{l \to 0} \frac{\ln N(l)}{\ln l}, \tag{13}$$

where $N(l)$ is the coverage of the set under investigation by open l-spheres.

It can be shown that $d_F \leq d_T$ and, if $d_F < d_T$, then the object is called a fractal object, i.e.

having a fractional dimensionality. Note that definition (13) has a purely geometric nature.

A set of events recorded in an experiment fill momentum space very nonuniformly, reflecting via the structure the dynamic mechanisms of particle production. That is why the event distribution over $N(l)$ bins will be highly non-uniform and this non-uniformity with a physical meaning is not reflected at all.

To generalize the notion (13), it is necessary to choose a universal measure fit to characterize the momentum-space structure non-uniformities. The subject of measure was discussed for dynamical systems turning to chaos [16]. For such systems, due to the necessity for transition from time averages to spatial ones, invariance of measure is required. There is no such problem for experimental data analysis, since the object (a population of points) can be considered as given, and time is not an essential characteristic. Besides, the object is compact: for any open coverage there exists a finite subcoverage.

Let us consider the l-coverage of the compact. In each bin $N_i(l)$ determine probability (cellular) measure (mass),

$$P_i(l) = \int_\Lambda \mathrm{d}\rho(x), \tag{14}$$

where Λ is the volume of a bin with size l, $\rho(x)$ is a probability density function determined in the whole space by means of some non-parametric method, by the experimental data or by a Monte Carlo simulation program [17].

From the point of view of experimental resolution it is important to use the cellular measure $P_i(l)$, though l should not be so small that the integral $\int_\Lambda \rho(x)$ loses its meaning.

The basic approach to dimensionality analysis lies in characterizing physical systems by the invariant probability measure singularities [18]. To do this, let us determine the scaling of the moments of the random quantity $p_i(l)$ of order q at scale l:

$$C_q(l) \equiv \langle p_i(l)^q \rangle \equiv \sum_{i=1}^{N(l)} p_i(l)^{q+1} \sim l^{\phi(q)},$$
$$\phi(q) = q d_{q+1}, \tag{15}$$

where d_q are the Renyi dimensions (generalized dimensions) determined for $-\infty < q < +\infty$. At $q=-1$, the relation (15) determines the capacity dimension $d_\mathrm{F} = d_0$, at $q=0$ the information dimensionality d_1, and at $q=1$ the correlation dimension d_2.

If the fractal is uniform (geometric), then

$$p_i \equiv p = 1/N_l, \quad N_l = N(l), \tag{16}$$

and

$$(1/N_l)^{q+1} N_l \sim l^{q d_{q+1}}, \tag{17}$$

hence we obtain for all q,

$$\ln N_l \sim -d_0 \ln l, \tag{18}$$

i.e. for uniform fractals the Renyi dimensions of any order are the same and are equal to the fractal dimension, and the scaling of the qth order momentum is characterized by the index $q d_0$, which increases linearly with the momentum order. And if the fractal is non-uniform, then all d_q are different (anomalous scaling) and the deviation from the dimensionality can be characterized by:

$$d_q - q d_0. \tag{19}$$

Thus, as in the case of normalized moments (6), the Renyi dimensions can serve as quantitative power indices of non-uniformity of both the rapidity distribution and the hadron distribution in momentum space.

The Renyi dimensions are defined as a slope connecting some values of $\{l_i\}$ with the corresponding values of $\{C_q(l_i)\}$ in a double-logarithmic scale. But the direct application of formula (15) to Renyi dimension calculation is rather time-consuming and, moreover, there are no instructions regarding the choice of the box-size sequence $\{l_i\}$. Algorithms based on nearest-neighbour information (NN-algorithms) are much more efficient than box-counting algorithms and they introduce a natural scale, the sample-averaged distance to NN,

$$\bar{R}_k, \quad k = 1, 2, \ldots, M,$$

where M is total number of events in the sample.

Using the ergodic theorem one can make a replacement [19,20],

$$\sum_{i=1}^{N(l)} p_i(l)^{q+1} \approx \sum_{j=1}^{M} \tilde{p}_j^q \approx Q_l, \tag{20}$$

where $\tilde{p}_j$ is the probability to find the point of the studied set not in the box of size l but inside the hypersphere of radius l, centered at some other point of the studied set and Q_l is the total number of q-tuples within this sphere.

For a $\overline{R}_k$ sequence the scaling relation takes the form

$$Q_{\overline{R}_k} \sim \overline{R}_k^{\phi(q)}. \tag{21}$$

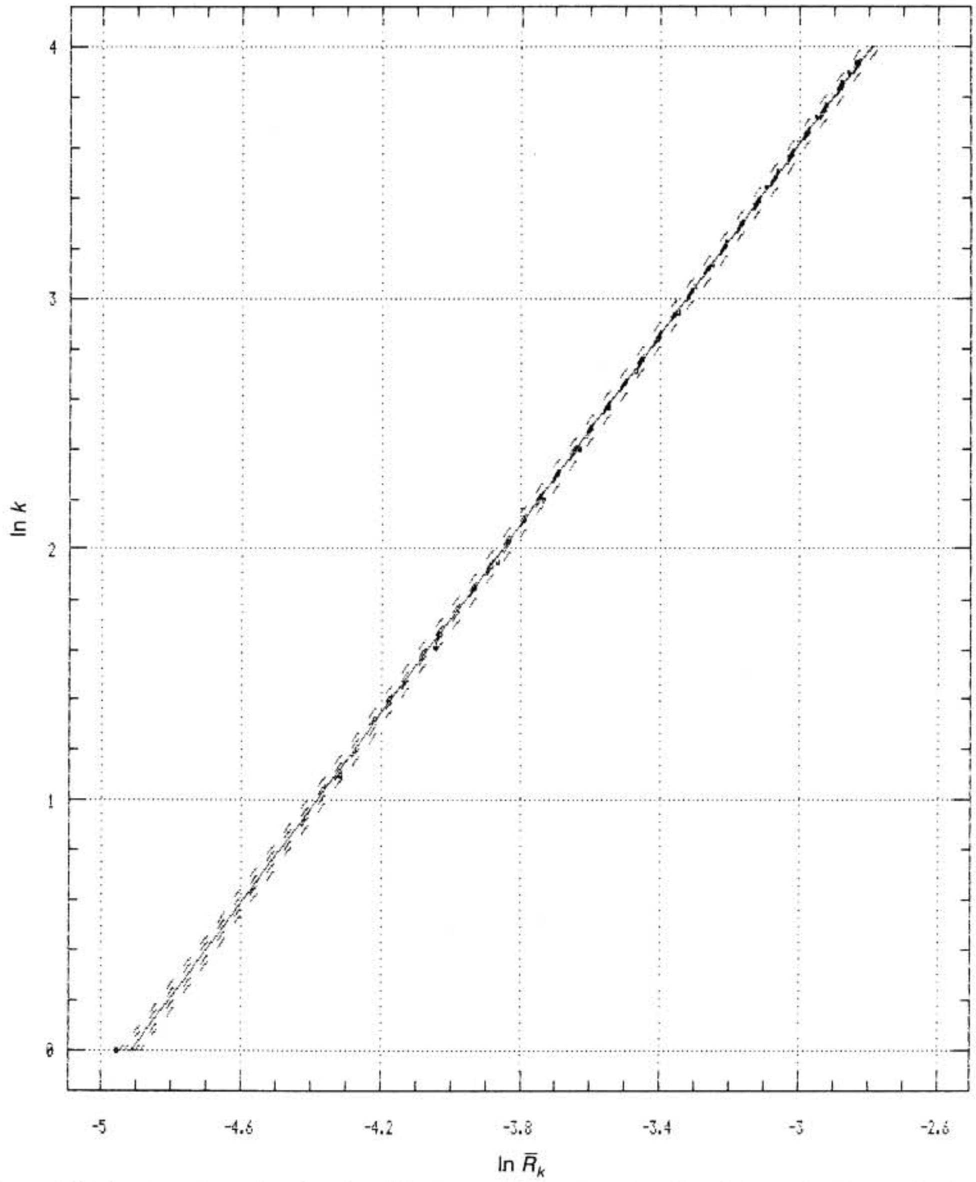

Fig. 2. The straight line slope determination, by which the correlation dimensionality of the Serpinski carpet is determined.

For $q = 1$ (correlation dimension), the number of q-tuples is simply equal to the number of sample events within l-spheres, and the left-hand side of (21) is equivalent to the mean number of sample points inside a hypersphere with radius equal to the average distance to the kth neighbour, i.e. is equal to the number k, so

$$k \sim \overline{R}_k^{d_2}. \tag{22}$$

Hence, the modified algorithm defines d_2 as a slope of the k-dependence of $\overline{R}_k$ in a double-logarithmic scale.

Figure 2 shows such dependence used to define the correlation dimension of the Sterpinski carpet. The dimension was determined by the least-squares method through 25 points: The logarithm of the number of the nearest neighbour 1, 3, ... 49 is approximated by the logarithm of the sample-averaged distance to the nearest neighbour. Of course, the number of events must be large enough; there is a definite relation between the space dimensionality and the minimum number of events needed to draw consistent conclusions.

By the $\phi(q)$ dependence it is possible to classify different events of multiparticle production [21], since a multifractal object can be considered as an interwoven family of uniform fractals, each obeying the scaling law with index d_0^α.

Note that the dimensionalities of d_0^α are not in any way connected with the regions where singularities of the probability measure arise, i.e. it is impossible to recover the spatial structure of the multifractal support from the d_q spectrum. That is why we believe that the local dimensionality

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.90|I2+45225+4 23 24336324 4242624+2 24554+4+4+332733+2 5552 26232255
.88|I4335 5+343+25+536++ 3342+45+43  34355++225+4422253435 46544+233
.85|I 5+  5372+ ++263+  2+4+3+ 625+42 33+6+  52442+++ 4223325232 2++
.83|I4 332+242424 +322354+ 33+43+2+633++23223223223 43532624+4+++ 3 6
.81|I+2522+5  + +  +42 +++84225443  +2 +++5233444233322 +  +2 26+3+2+
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.76|I322 332       24    2233  63+       33  423 4  +42       34+ +25
.73|I32 456+       23242553+23+ 42      233223535323322       35+ 232
.71|I4+2532+  4322+3++237243+546+3 2+3+ +24 3475+5+6 27+24 33425523+3
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.14|I44+ 237       +62 +52+23+55+       3+43+5+432  +22        2 222 2
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.09|I5++5+3 +4+++++6+23483+ 4++33533433+++32+5+422+2 ++5232422+56+2+4
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.05|I225+24+373+  63633+4644+++4345+++32443+343354  +2+46+++66633 36+
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     .00            .22             .45             .67            .9
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Fig. 3. The 5th generation of the Serpinski carpet, 5000 points.

introduced in ref. [22] may be useful in separating the momentum-space regions where considerable fluctuations of the invariant probability measure are observed.

A description of the algorithm for the local and global correlation dimension calculation is presented in the next section, along with an interesting relation of the fractal dimensions to the *intrinsic dimension*, a notion developed also in the mathematical theory of pattern recognition.

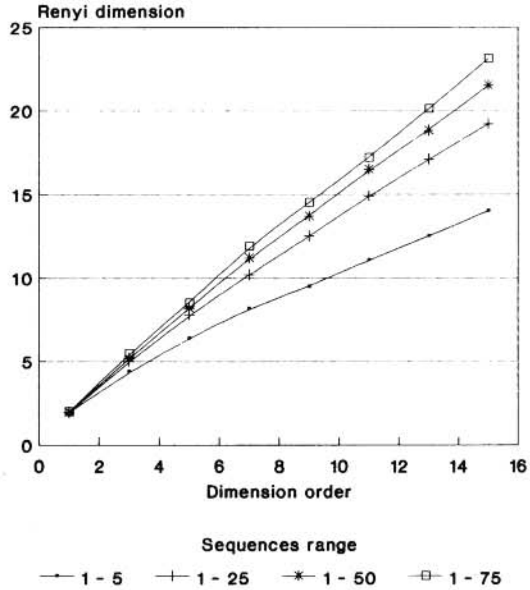

Fig. 5. Comparison of the degree of non-uniformity of the population of a unit square by two-dimensional random numbers (the RNDM generator). The narrower the $\bar{R}_k$ sequence for determination of the Renyi dimension, the higher the nonuniformity.

3. KNN estimation of probability density. Local and global dimensionality.

Consider the KNN estimation of probability density [23] which is a development of the well-known histogram method,

$$\rho_k(x_i) = \frac{k}{MV_k(x_i)}, \qquad (23)$$

where $V_k(x_i)$ is the volume of a d-dimensional hypersphere containing the k nearest neighbours to x_i.

$$V_k(x_i) = V_d R_k^d, \qquad V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}, \qquad (24)$$

where R_k is the distance to the kth nearest neighbour of x_i and $\Gamma(z)$ is the gamma function. From eqs. (23) and (24) we can readily obtain [24]

$$\ln R_k(x_i) = \frac{1}{d}\ln k + \ln[MV_d\rho_k(x_i)]^{-1/d}. \quad (25)$$

Equation (25) cannot be solved for d, since the estimate of $\rho(x_i)$, as one can see from eq. (23), depends on k. Therefore, let us average R_k over the whole sample, according to the distribution function,

$$f_{k,x}(R) = CdR^{d-1}\frac{(CR^d)^{k-1}}{\Gamma(k)}e^{-CR^d}, \qquad (26)$$

where $C = M\rho(x)\,V_d$.

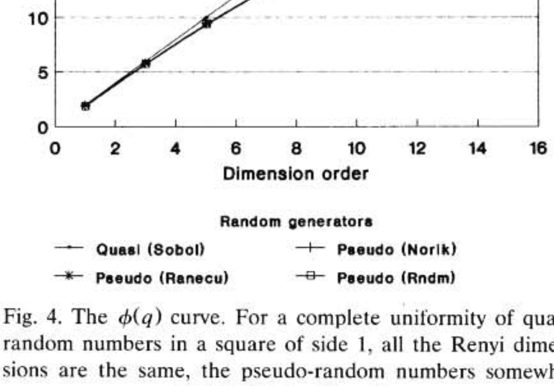

Fig. 4. The $\phi(q)$ curve. For a complete uniformity of quasi-random numbers in a square of side 1, all the Renyi dimensions are the same, the pseudo-random numbers somewhat deviate from uniformity.

In the approximation of small R and large M we obtain the following equations:

$$\ln G_{k,d} + \ln \overline{R}_k = \frac{1}{d} \ln k + \text{constant}, \tag{27}$$

$$G_{k,d} = \frac{k^{1/d}\Gamma(k)}{\Gamma(k+1/d)}, \tag{28}$$

where $\overline{R}_k$ is the sample-averaged distance to the kth nearest neighbour and the constant is independent of k.

The difference between this scaling equation and those we obtained previously by a completely different approach, consists in the so-called iterative addition $G_{k,d}$, which is close to zero for all k and d. Therefore, we solve this equation iteratively, first assuming $G_{k,d} = 0$, and then, having obtained d_i, we calculate G_{k,d_i} and determine the value of d_{i+1}. We stop the iterations when d becomes nearly constant.

Such verification of d-estimates is connected with the averaging of the correlation integral. The correlation integral (the number of sample points inside a hypersphere of fixed radius) is a random variable belonging to a binomial distribution with parameter $\rho(x)$ (the probability for the sample point to fall within this hypersphere).

Notice that our estimate is global, i.e. the whole sample is characterized by one number, though local differences are possible. From this point of view, local dimensionality is much more interesting, since it allows us to detect local inhomogeneities corresponding to various dynamical mechanisms.

Consider eq. (25) again. Apart from sample averaging, there is also one more way to get a linear equation for determining the dimension. For this, one must choose the series $\{k_j\}$ such that the density estimates are very close, and hence the dependence of $\rho_k(x)$ on k can be ignored. Following these chosen values $\{k_j\}$ and the corresponding $\{R_{k_j}(x_i)\}$, one can estimate the local dimension at the point x_i.

4. The simulation study

The Renyi dimension was determined for the samples generated by the algorithm for the Serpinski carpet (fig. 3), the Henon map, and for

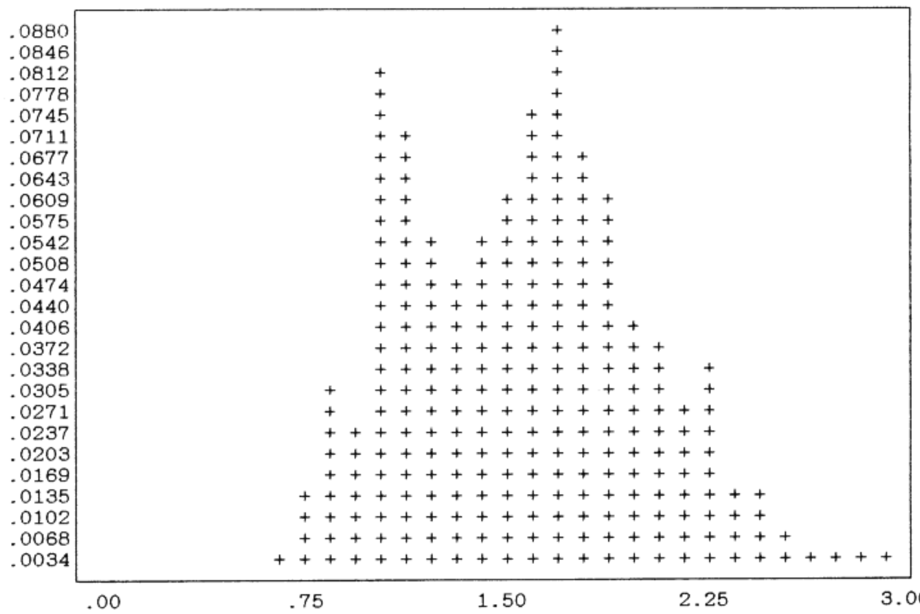

Fig. 6. Histogram of the local dimensionality of a mixed sample – a Serpinski carpet and a Henon map.

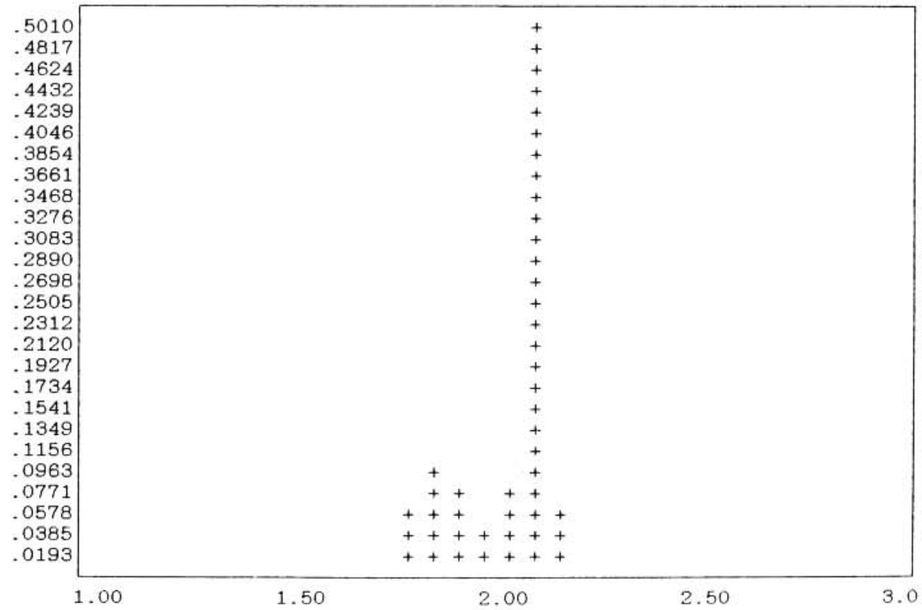

Fig. 7. Distribution (normalized histogram) of the local dimensionalities determined for samples of two-dimensional quasi-random numbers.

samples obtained by different random-number generators.

The experiments were carried out to investigate the sensitivity of the method to the choice of parameters which include the sample size, the sequence of the nearest neighbours and the order

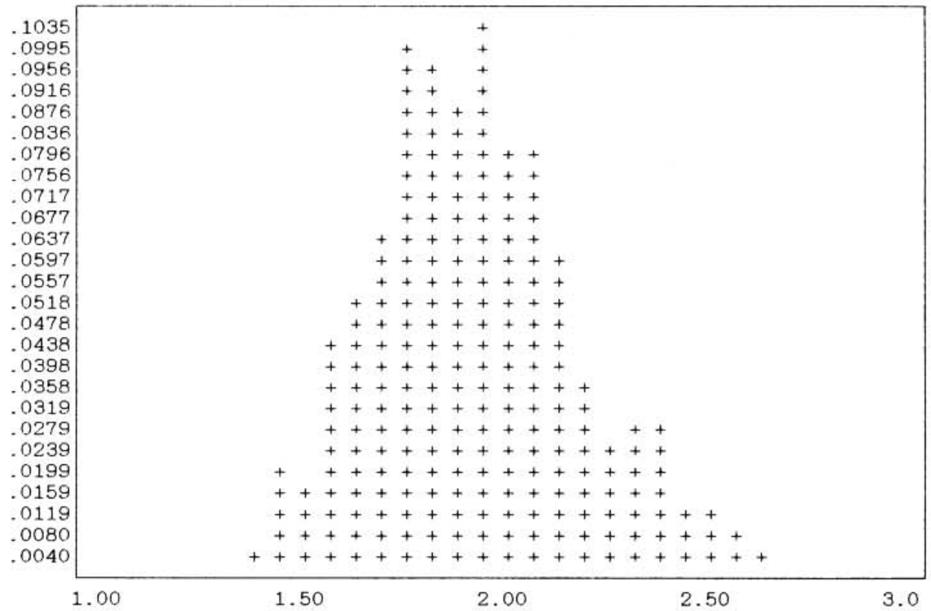

Fig. 8. Distribution (normalized histogram) of the local dimensionalities determined for samples of two-dimensional pseudo-random numbers.

of the Renyi dimensions, and to study the possibilities of separating the regions with anomalous structure. We also considered the quality of the quasi-random-number generators, an important aspect for many applications. For comparison of the uniformity of the population of an N-dimensional space by "random" numbers, we used "quasi-random" numbers – LP-sieves, which uniformly fill an N-dimensional cube [25].

Figure 4 presents the Renyi dimensions of order from 1 to 15 – the function $\phi(q)$. The three random-number generators being compared are: RNDM, which has been widely used in the past decade; RANECU, a generator recently recommended by F. James [26], and NORIK, a matrix generator designed in the Yerevan Physics Institute [27].

Sets of two-dimensional random quantities distributed in a square of side 1 were considered. The slopes connecting the values of the moments of the invariant probability measure (15) were calculated through 70 points for distances equal to the average distance to the nearest neighbours with numbers from 6 to 75, the orders of dimensions being chosen from 1 to 15. The sample sizes were 1000 and 5000.

For a strictly periodic structure of LP-sieves, all the Renyi dimensions are the same: $\phi(q) = qd_0$ and the random-number generators show some deviation from uniformity, which is due to the limited sample sizes. The matrix generator reveals somewhat better results.

Figure 5 presents Renyi dimensions calculated using different $\bar{R}_k$-sequences (the sequences con-

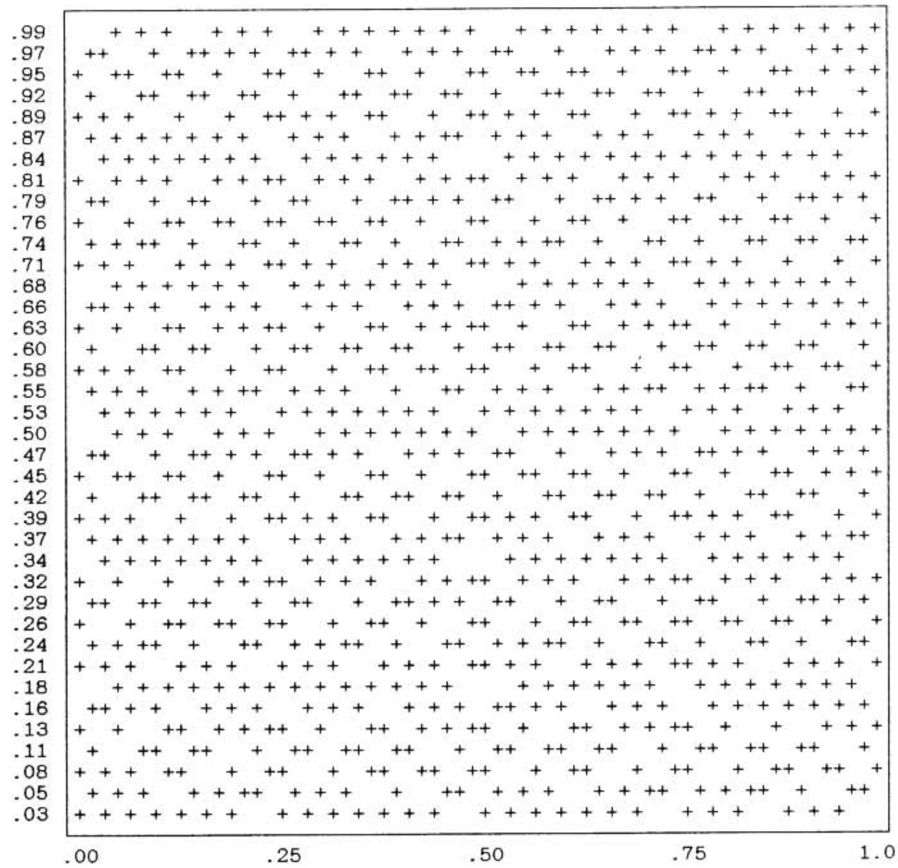

Fig. 9. A planar LP-sieve, 1024 nodes.

sisted of average distances from 1 to 5, 1 to 25,...,1 to 75 nearest neighbours). The smaller the range over which the dimension is determined, the more the random fluctuations and the more the difference between the function $\phi(q)$ and the line $y = qd_0$, which corresponds to complete uniformity.

Figure 6 shows the histogram of the local dimensions of a mixed sample consisting of a mixture of 500 events of Serpinski's carpet ($d_2 \approx 1.9$) and 500 events of Henon's map ($d_2 \approx 1.2$). Two peaks are clearly seen, which correspond to two modes (the correlation dimensionality is binned).

Unimodal distributions corresponding to data of the same type are shown in figs. 7 and 8.

A quasi-periodical distribution was used to "scan" the fractal support with the purpose of determining the anomalous areas: the dimensionality was calculated in the nodes of the LP-sieve (fig. 9). Figure 10 presents the results of scanning a square of side 0.9 where the Serpinski carpet is situated. For the sieve points falling into the empty areas of the carpet the fractal dimension turned out to be greater than 2.2, which allows them to be reliably separated.

The quasi-random sequence itself also turned out to be non-uniform on the boundaries of its support shown in fig. 11.

The program code is written in Fortran 77 for VAX and IBM-compatible computers operating under VM. Some subroutines from the KNN

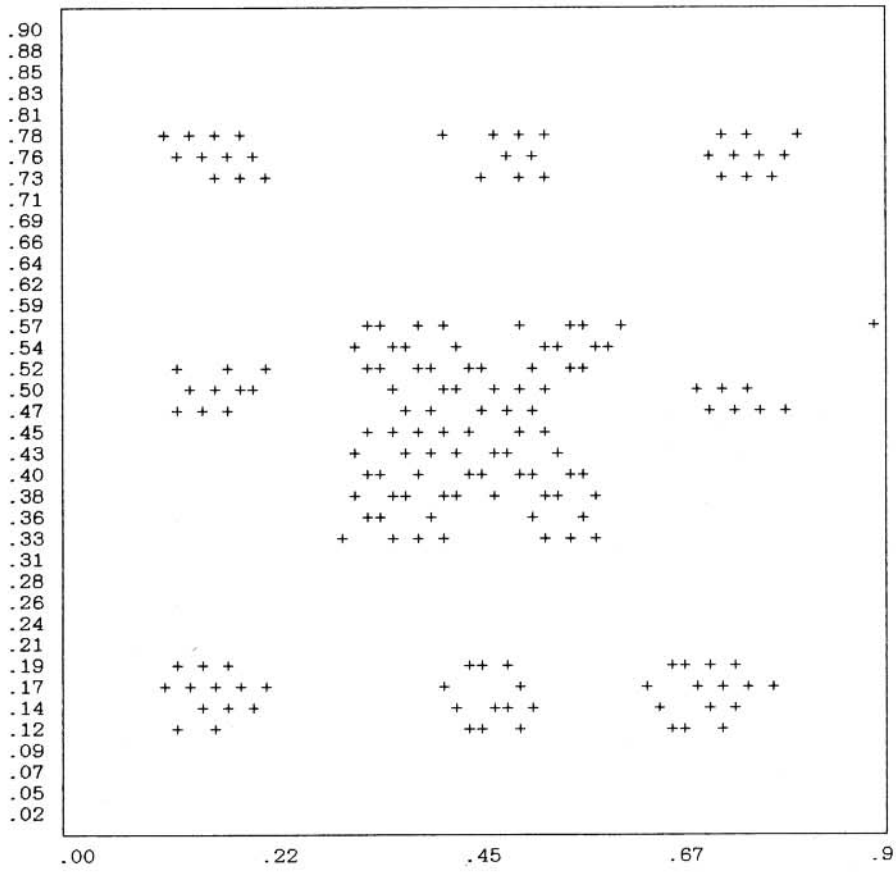

Fig. 10. The results of scanning a Serpinski carpet to denote points where the local dimension is larger than 2.2.

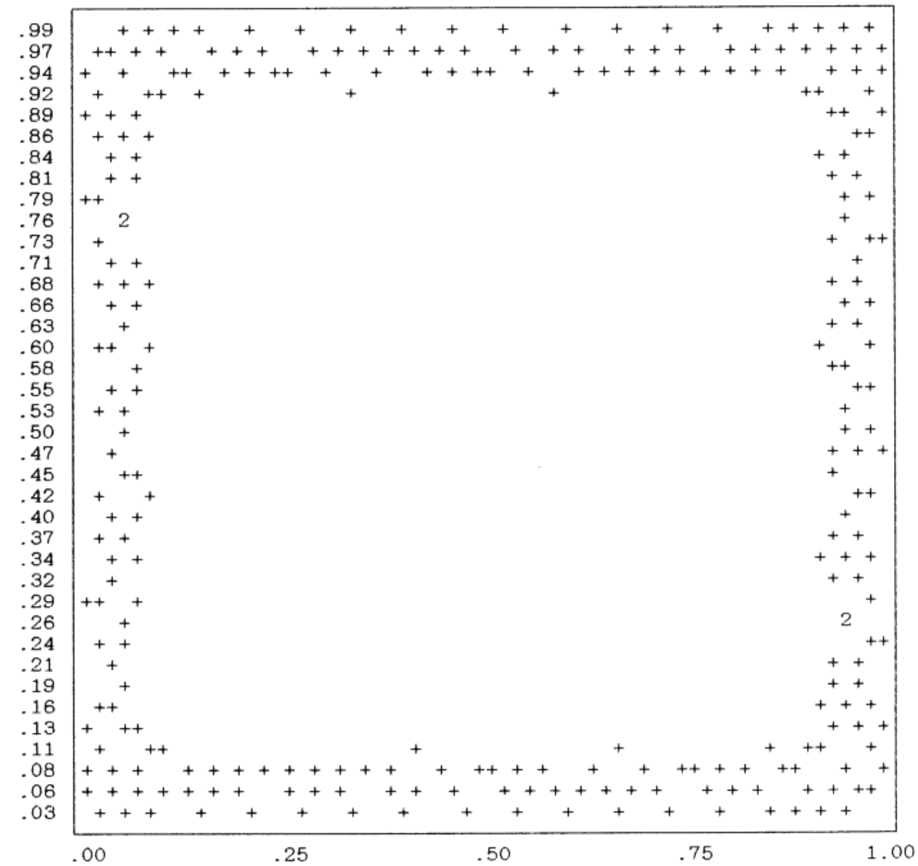

Fig. 11. The results of scanning a planar LP-sieve over the boundaries of its support; + denotes points where the local dimension is larger than 2.2.

multivariate density-estimation package [17] are used for NN distance calculations and q-tuple counting. The calculations have been carried out on an EC-1046 computer in the computation center of the Yerevan Physics Institute.

5. Conclusion

To summarise, we have investigated a new method of multiparticle data analysis which can deal with the large number of particles produced in modern colliders.

We have demonstrated how the Renyi dimensions can be used as a quantitative measure to outline possible inhomogenieties in a $3N$-dimensional momentum space or in the rapidity (pseudo-rapidity) distributions.

We introduce a simple technique for Renyi dimension calculation. A universal scale – sample-averaged distance to NN – is offered. A q-tuple counting algorithm provides an evaluation of Renyi dimensions in a sizeable range of values of q. The KNN algorithm for calculating the correlation dimension is much more suitable and precise than box-counting algorithms.

By the local dimension distribution obtained on fractal support we can judge the relative importance of the different mechanisms taking part in the creation of the data.

The application of these ideas to the analysis of multiparticle production dynamics requires intensive Monte Carlo simulations and detailed quantitative comparisons of simulated and experimental data.

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